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ON SEMI- STRONG (WEAK)CJ-TOPLOGICAL SPACES

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ABSTRACT

In this paper we introduced new types of spaces as Weak CJ-space, Semi- strong CJ-space and Semi-Weak CJ-space, also we studied the relationship between them and the relation of them withCJ-space and strong CJ-space.

Keywords- *CJ- space, strong CJ-space, Semi- Strong CJ- space, Weak CJ-space and Semi- Weak CJ-space.*

I. INTRODUCTION

Consider the concepts of compact space and countably compact space, a topological space X is compact if every open cover has a finite subcover (This is equivalent to "closed" and "bounded" in Euclidean space) [1]. A topological space X is countably compact if every countable open cover has a finite subcover (This is equivalent to every infinite set having a cluster point)[2]. In [3] E.Michael introduced the concepts of J-space and Strong J-space.

A topological space X is a J-space if, whenever {A, B} is a closed cover of X with A$\cap$ B compact, then A or B is

compact. A space X is a Strong J-space if every compact K$\subset$ X is contained in a compact L$\subset$ X with X\L is

connected. Michael also introduced three classes of spaces which are closely related to J-space and Strong J-space, these spaces are Sime-Strong J-space, Weak J-space and Sime-Weak J-space.

In [4] we introduced the concepts of CJ-space and strong CJ-space by replacing the term compact in the definitions of J-space and Strong J-space, with the term countably compact. In this paper we introduced three types of spaces which are closely linked to CJ-space and Strong CJ-space.

II. SEMI- STRONG CJ- SPACE, WEAK CJ-SPACEAND SEMI- WEAK CJ- SPACE

In this section we define some concept which is important and necessary to get the aim of this paper such Semi-Strong CJ- Space, Weak CJ-space and Semi- Weak CJ- space.

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Definition 1.1:A topological spaceis a Semi- Strong CJ- space if for every countably compact $K\subset X$ there is a countably compact subset L of X such that $K\subset L$ and there exists a connected subset C of X with $C\subset X\setminus K$ and $C\cup L=X$.

Definition 1.2:A topological space is a Weak CJ-space if, whenever {A,B,C}is a closed covering of X with Kcountably compact and $A\cap B=\emptyset$, then A or B is countably compact.

Definition 1.3: A topological space is a Semi- Weak CJ-space if, whenever A and B are disjoint closed subsets of X with countably compact boundaries, then A or B is countably compact.

Theorem 1.4:[4] Let X be any topological space , then the following conditions are equivalent:

1. X is a CJ-space,
2. For any $A\subset X$ with countably compact boundary , cl(A) or cl(X\A) is countably compact,
3. If A and B are disjoint closed subsets of X with ∂A or ∂B countably compact , then A or B is countably compact.

Theorem 1.5: Consider the following properties of a topological space ,

a) X is a Strong CJ- space.
b) X is a Semi- Strong CJ- space.
c) X is a CJ- space.
d) X is a Semi- Weak CJ- space.
e) X is a Weak CJ-space.

Then (a)$\Longrightarrow$(b)$\Longrightarrow$(c)$\Longrightarrow$(d)$\Longrightarrow$(e)

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Proof: (a)⟹(b)

Let X be a Strong CJ- space and let K⊂X be a countably compact, then there exists a countably compact subset L of X such that K⊂L and X\L is connected by definition of Strong CJ-space. Now let C= X\L, then C is connected and C⊂ X\K since K⊂ L, and C⋃L= X. Hence X is a Semi- Strong CJ- space.

(b) ⟹ (c)

Let X be a Semi- Strong CJ- space and let {A,B} be a closed cover of X with A⋂B countably compact, so there exists a countably compact L⊂X such that A⋂B⊂L and there exists a connected subset C of X with C⊂ X\ A⋂B and C⋃L= X by definition of Semi- Strong CJ- space. Note that (A⋂C) ⋂ (B⋂C) = (A⋂B) ⋂ C = ∅ since C⊂ X\ A⋂B, and that (A⋂C) ⋃ (B⋂C) = (A⋃B) ⋂ C = X⋂C = C,so we get a disjoint closed cover {A⋂C, B⋂C} of Cwhich is connected, therefor C must be in A⋂C or in B⋂C, and thus C⊂ A or C⊂B. If C⊂ A, then C⋂B=∅, it

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follows that B⊂ X\C⊂ L which is countably compact, so B is countably compact. Similarly if C⊂ B, then A is countably compact. Hence X is CJ- space.

(c) ⟹ (d)

Let X be any CJ- space and let A, B be tow disjoint closed subsets of X with countably compact boundaries, then A or B is countably compact by Theorem 1.4. Thus X is Semi-Weak CJ-space.

(d) ⟹ (e)

Assume that X is a Semi-Weak CJ- space and let {A, B, C} be a closed cover of X with K countably compact and A⋂B=∅. Note that, so

, so∂A⊂ K⋂A⊂ K, similarly we can prove that ∂B⊂ K, and thus ∂A and ∂B are countably compact, it follows by (d) that A or B is countably compact. Hence X is Weak CJ- space.

Remark 1.6: A Semi- Strong CJ-space need not be Strong CJ- space.

For example: Let us take the usual topological space joining (n,0) to (n+1,1/i). Let . Then Y is not Strong CJ-space for if K⊂Y is countably compact, then Y\K is not connected. But Y is Semi- Strong CJ-space, to prove that let. Note that is countably compact and is connected and = Y for each n. Now let K be a countably compact subset of Y and pick n such that, then .

Remark 1.7: A CJ- space need not be Semi- Strong CJ-space.

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For example: Consider the Odd-Even topology defined on the set of natural numbers$\mathbb{N}$ this topology is generated by the partition P ={{2k-1,2k}; k∈ℕ}. The only countably compact subsets of ℕ are the finite subsets, so if we take a closed cover {A,B} of ℕ with A⋂B countably compact , that is mean A⋂B is finite set and since the intersection of any two infinite sets in this space must be an infinite set, so A or B must be finite , that is mean A or B is countably compact . Hence ℕ is CJ-space.

Butℕ is not Semi- Strong CJ-space since every countably compact subset of ℕ is finite and every infinite subset of ℕ is non- connected, so if we take a countably compact subset K of ℕ and a countably compact subset L of ℕ such that K⊂ L and a connected subset C of ℕ with C⊂ℕ\K and C∪ L= ℕ, then C must be infinite which is a contradiction.

Theorem 1.8 [4]: In any metric space (X, d), the following concepts are equivalent:

1. X is CJ-space.
2. X is Strong CJ-space.
3. X is J-space.
4. X is Strong J-space.

Proposition 1.9 [3]: If are connected and non- compact, then is a Strong J- space.

Proposition 1.10: If X is a CJ- space and, then Z is a Semi- Weak CJ- space.

Proof: Let A, B be two disjoint closed subsets of Z with countably compact boundaries, then A or B. Suppose that B and let E= cl(X\B), then {B,E} is a closed cover of X with E⋂ B= ∂B which is countably compact, so B or E is countably compact since X is CJ- space. But A⊂ E⋃, so A or B is countably compact, and thus X is a Semi- Weak CJ- space.

Remark 1.11: A Semi- Weak CJ- space need not be CJ- space.

For example: Let. Then X is a CJ- space by Proposition 1.9 and Theorem 1.8, so Z is a Semi- Weak CJ- space by Proposition 1.10. To see that Z is not a CJ- space, let . Then {A,B} is a closed cover of Z with A⋂B = the closed segment joining (0,0) to (0,1) which is countably compact, but neither A nor B is countably compact.

Proposition 1.12: Let be a closed cover of a topological space X with non- countably compact. If are Weak CJ- spaces, then so is X.

Proof: Let {A,B,K} be a closed cover of X with A⋂ B = ∅and K is countably compact. To prove Aor B is countably compact, let , for i=1, 2. Then is a closed cover of is countably compact. Now by using the fact saying that is Weak CJ- space, we get is countably compact. Suppose that is countably compact, we claim that is also countably compact, for if is not countably compact, so must be countably compact since is Weak CJ- space, it follows that is countably compact, but is a closed subset of C, so must be countably compact which is a contradiction. Thus is countably compact. Similarly we can prove that A is countably compact whenever is countably compact.

Remark 1.13: A Weak CJ- space need not be Semi- Weak CJ- space.

For example: Let. To see that Z is a Weak CJ- space, let , then is a closed cover of Z, and which is non- countably compact, but are both Semi- Weak CJ- space since they are homeomorphic to the space Z of Remark 1.11, and thus they are Weak CJ- space.Hence Z is Weak CJ- space by Proposition 1.12. To see that Z is not a Semi- Weak CJ- space, let , then A and B are disjoint closed subsets of Z With countably compact boundaries, but neither is A nor B countably compact.

Proposition 1.14: Let be a closed cover of a topological space X with non- countably compact. If are Semi- Strong CJ- spaces, then so is X.

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Proof: Let K⊂ X be a countably compact and let , thenis a closed subset of K, and thus countably compact subset of

the Semi- Strong CJ- space , so there exists a countably compact subset of such that and there exists a connected subset of such that (for i=1, 2),by definition of Semi- Strong CJ- space. Now let , so L is a countably compact

subset of X with K⊂ L and C⋃L=X and C⊂ X\K. It remains to show that C is connected, we need only cheek that

since are connected. Note that, for if , then is a closed subset of L which is countably compact, so is countably compact which is a contradiction. Also we have, and thus . Hence is connected. Therefor X is a Semi- Strong CJ-space.

Definition 1.15: A map is said to be countably perfect if it is closed and is countably compact subset of X for every countably compact subset B of Y.

Definition 1.16: A map is said to be boundary- countably perfect if it is closed and is countably compact subset of

X for every y ∈Y.

Theorem 1.17: A topological space (X, τ) is CJ-space if and only if every closed boundary- countably perfect map from X onto a non-countably compact space Y is countably perfect.

Proof: The "if" part

Suppose that is a CJ-space and is a non- countably compact and is a closed boundary- countably perfect map. We

have to show that f is countably perfect, let y ∈Y, then is a subset of the CJ-space X with countably compact

boundary, it follows by Theorem 1.13 that either is countably compact. But is not countably compact, for if is

countably compact, then Y= {y} ⋃f(is countably compact which is a contradiction. Hence is countably compact.

The "only if" part

Suppose that every closed boundary- countably perfect map from X onto a non-countably compact space Y is

countably perfect. To prove that X is CJ-space, let {A, B} be a closed cover of X with A⋂B countably compact. Let

Y=X/B and f:X→Y be the quotient map and let , then f is boundary- countably perfect since it is closed and for each

y∈ Y, is countably compact because it is either a singleton (for y≠) or a closed subset of A⋂B (for y=)

Now if Y is non-countably compact, then f is countably perfect by hypothesis and thus B = is countably compact.

If Y is countably compact, so f(A) is countably compact since it is closed subset of Y. On the other hand we have is

countably perfect since it is closed map and its fibers are either singletons or equal to A⋂B. Hence A= is countably

compact.

Proposition 1.18: The following properties of a space X are equivalent:

a) X is a Semi- Weak CJ-space

b) Iff:X→Y is boundary- countably perfect, then is non- countably compact for at most one y∈ Y.

Proof:(a) ⟹ (b)

Suppose that X is a Semi-Weak CJ-space and in Y, and let (for i=1, 2). Then and are countably compact since f is boundary- countably perfect, so is countably compact by definition of Semi- Weak CJ- space.

(b) ⟹ (a)

Suppose with countably compact boundaries. Define a relation R on X such that .Then .

Let Y be the quotient space of X with respect to the relation R, and let f:X→Y be the quotient map, so f is a closed, continuous and onto map. Now to show that f is boundary- countably perfect, it is sufficient to prove that is

countably compact for each y∈Y. Let y∈Y, then .

But are countably compact by hypothesis and ∂ {y} is also countably compact, so f is boundary- countably perfect

and hence is countably compact by (b).

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